12 A Visit to Valuation and Pseudo-Valuation Domains

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INTRODUCTION

Throughout this paper, R denotes a commutative domain with $1\neq 0$ and K denotes the quotient field of R. Recall [9], a prime ideal P of R is called strongly prime if $x,y\in K$ and $xy\in P$ imply that $x\in P$ or $y\in P$. If every prime ideal of R is strongly prime, then R is called a Pseudo-Valuation Domain (abbreviated PVD). In this paper, we give alternative proofs of some well-known results in [2], [9]. Let P be a nonzero strongly prime ideal of R. If P contains a prime element of R, then we show that P is a principal maximal ideal of R and R is a valuation domain. Furthermore, we give an alternative proof of the fact [2, Proposition 4.3] that $P^{-1}=(P:P)=\{ x\in K : xP\subset P \}$ is a ring and we give a more general version of this fact.

Part of the following result appeared in [9, Corollary 1.3] and a stronger version appeared in [2, Proposition 4.2]. But the proof we give here is somewhat different from those in [9] and [2].

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PROPOSITION 1. Let P be a strongly prime ideal of R and I be an ideal of R. Then P⊂I or I⊂P, that is, P and I are comparable. In particular, if R is a PVD, then the prime ideals of R are linearly ordered and therefore R is quasi-local.

Proof: Deny. Then there exist $i \in I$ and $p \in P$ such that $i \notin P$ and $p \notin I$. But $(p/i)i = p \in P$ and $(p/i) \notin P$ and $i \notin P$, a contradiction, since P is strongly prime. The remaining part of the Proposition is now clear. ■

The following Proposition can be proved using [2, Propositions 4.2 and 4.8]. We give a proof of it that relies on the above Proposition and the definition of strongly prime ideals.

PROPOSITION 2. A domain R is a PVD if and only if a maximal ideal of R is strongly prime.

Proof: We only need to prove the converse. Let M be a maximal ideal of R that is strongly prime. By the first part of the above Proposition, we conclude that R is quasi-local and M is the maximal ideal of R. Let P be a prime ideal of R and $x, y \in K$ and $xy \in P$. If x and y are in R, then $x \in P$ or $y \in P$. Hence, suppose $x \notin R$. Since $xy \in M$ and $x \notin R$, we have $y \in M$. Suppose $y \notin P$. Then y^2 is not in P and therefore $d = (y^2/xy) \notin R$. But $dx = y \in M$ and neither x nor d is in M, a contradiction. Thus, $y \in P$ and P is strongly prime. ■

The following proposition was first proved in [9, Proposition 2.2]. The proof in [9] depends upon [13, Theorem 1] that a GCD-domain whose primes are linearly ordered must be a valuation domain. For other proofs, often of more general statements see [4, Corollary 4.3], [14, Corollary 3.8], and [15, Proposition A]. Yet another proof of it was given in [3, Corollary A.5]. The proof in [3] relies on [3, Corollary A.4 and Proposition 2.3]. Now, we give a proof of it that depends upon the definition of strongly prime ideals and some basic knowledge of GCD-domains.

PROPOSITION 3. A domain R is a valuation domain if and only if it is both a GCD-domain and a PVD.

Proof: We only need to prove the converse. Suppose R is both a GCD-domain and a PVD. Let M be the maximal ideal of R and a,b be nonzero nonunit elements of R. Suppose g.c.d(a,b)=d such that d is associated to neither a nor b. Let $m=a/d$ and $n=b/d$. Then neither m nor n is a unit in R. It is well-known [12, Theorem 49, P. 32] that g.c.d$(m,n)=1$ and g.c.d$(m,n^2)=1$. Hence, $g=(m/n)\notin R$ and $h=(n^2/m)\notin R$. But $gh=n\in M$, a contradiction, since neither g nor h is in M. ■

Now, we state the following result :

PROPOSITION 4. Let P be a nonzero strongly prime ideal of R. If P contains a prime element of R, then P is a principal maximal ideal of R.

Proof: Suppose that P is nonmaximal. Then there exists a nonunit element x in R such that $x\notin P$. Let $p\in P$ such that p is a prime element of R. By Proposition 1, we have $P\subset(x)$. In particular, $p\in(x)$, a contradiction, since p is prime and $x\notin P$ and x is a nonunit element of R. Hence, P is a maximal ideal of R. We claim that $P=(p)$. Deny. Then there exists $y\in P$ such that $d=(y/p)\notin R$. Hence, $h=(p^2/y)\notin R$. (Observe that if $(p^2/y)\in R$, then either p divides y or y is a unit in R, and in both cases we have a contradiction.) But $dh=p\in P$, a contradiction. Thus, $P=(p)$. ■

COROLLARY 1. If P is a nonzero principal strongly prime ideal of R, then P is maximal.

It was shown [9, Corollary 2.9], that if R is a PVD and it has a nonzero principal prime ideal of R, then R is a valuation domain. The proof in [9] relies on [9, Proposition 2.8 and Lemma 1.6]. We give an alternative proof of this fact.

PROPOSITION 5. If a PVD R has a nonzero principal prime ideal, then R is a valuation domain.

Proof: Let $P=(p)$ for some prime p of R be a principal prime ideal of R. By Corollary 1, P is a maximal ideal of R. Let x,y be nonzero nonunit elements of R. Suppose $d=(x/y)\in K-R$. Let $h=(py/x)$. Since $dh=p\in P$ and $d\notin R$, we have $h\in P$. Thus, $py=xz$ where $z=h=(py/x)\in P$. Hence, p divides z and therefore x divides y. Thus, R is a valuation domain. ■

Remark: It is shown [6, Corollary 2.4] that a nonzero principal prime ideal of a going down domain (denoted GD-domain) is a maximal ideal. Since every PVD domain is divided (that is, for every prime ideal P of R, we have $P=PR_P$), see [6, section 4], and every divided domain is a GD-domain, see [5, Proposition 2.1], one may conclude that the principal prime ideal in the above Proposition is a maximal ideal of R.

A stronger version of [9, Corollary 2.9] is the following

COROLLARY 2. Suppose a domain R has a nonzero principal strongly prime ideal. Then R is a valuation domain.

Proof: By Proposition 2 and Corollary 1, R is a PVD. Hence, by Proposition 5, R is a valuation domain. ■

It was shown [2, Proposition 4.3] that if P is a nonprincipal strongly prime ideal of R, then $P^{-1}=\{ x\in K : xP\subset R \} = (P:P) = \{ x\in K : xP\subset P \}$ is a valuation domain. The proof in [2] that P^{-1} is a ring depends upon [2, Propositions 4.2, 2.5, and 3.3]. In the following Proposition, we give an alternate proof and a stronger version of this fact. Recall [5], a prime ideal P of R is called a divided prime if it is comparable to every principal ideal of R, that is, $PR_p=P$.

PROPOSITION 6. Suppose P is a nonprincipal divided prime ideal of R. Then $P^{-1} = (P:P)$ is a ring. In particular, if P is strongly prime and nonprincipal, then P^{-1} is a valuation domain.

Proof: let $x \in P^{-1}$. Suppose for some $p \in P$, $xp = d \in R-P$. Then $p/d \in R$, since $(p) \subset (d)$ by the hypothesis. Since $(p/d)d = p \in P$ and $d \notin P$ and P is prime and both p/d, d are elements of R, we have $p/d \in P$. But $x(p/d) = d/d = 1$ and therefore $P = x^{-1}R$ is principal, a contradiction. Hence, $d \in P$, a contradiction. Thus, $P^{-1} = (P:P)$ is a ring.

Suppose P is nonprincipal and strongly prime. Then P is comparable to every principal ideal of R by Proposition 1 or [2, Proposition 4.2]. Hence, P^{-1} is a ring. The proof that P^{-1} is a valuation domain is given in [2, Proposition 4.3]. ■

Anderson [2, Proposition 4.6] showed that for a nonzero ideal I, the following two statements are equivalent :

(1) I is a nonprincipal strongly prime ideal.

(2) I^{-1} is a ring and I is comparable to every principal fractional ideal of R.

We terminate our visit with the following Proposition :

PROPOSITION 7. The following statements are equivalent for a nonzero proper ideal I of R.

(1) I is a nonprincipal divided prime ideal.

(2) I^{-1} is a ring and I is comparable to every principal ideal of R.

Proof: (1) implies (2) is clear from Proposition 6 and the definition of divided prime. We only need show that (2) implies (1). Since I^{-1} is a ring, I is nonprincipal. Let $S = R-I$ and let $x, y \in S$. Since I is comparable to every principal ideal of R, 1/x and 1/y are elements of I^{-1}. Since I^{-1} is a ring, we have $(1/x)(1/y) = 1/(xy) \in I^{-1}$. Since I is nonprincipal and $1/(xy) \in I^{-1}$, we have $xy \in S$. Thus, S is a multiplicatively closed subset of R and therefore I is prime. ∎

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REFERENCES

1. Anderson, D. F.," Comparability of ideals and valuation overrings," Houston J. Math., 5 (1979), 451-463.
2. Anderson, D. F.," When the dual of an ideal is a ring," Houston J. Math., 9 (1983), 325-332.
3. Anderson, D. F., Dobbs, D. E.," Pairs of rings with the same prime ideals," Canad. J. Math., 32 (1980), 362-384.
4. Dawson, J., Dobbs, D., E., " On going down in polynomial rings," Canad. J. Math. 26 (1974), 177-184.
5. Dobbs, D. E.," Divided rings and going down, Pacific J. Math., 67 (1976), 353-363.
6. Dobbs, D. E.," Coherence, ascent of going down, and pseudo-valuation domains," Houston J. Math., 4(1978), 551-567.
7. Dobbs, D. E., Fontana, M., " Locally pseudo-valuation domains," Annali di Matematica pura ed applicata,(IV), Vol. CXXXIV (1983), 147-168.
8. Gilmer, R., Multiplicative ideal theory, Queen's Papers in pure and Applied Mathematics, Vol. 90, Kingston, Ontario (1992).
9. Hedstrom, J. R., Houston, E. G.," Pseudo-valuation domains," Pacific J. Math., 75 (1978), 137-147.
10. Hedstrom, J. R., Houston, E. G.," Pseudo-valuation domains,II," Houston J. Math., 4 (1978), 199-207.

11. Huckaba, J. A., Papick, I. J.," When the dual of an ideal is a ring," Manuscripta Math., 37 (1982), 67-85.

12. Kaplansky, I., Commutative rings, The Univ. of Chicago Press, Chicago, (1974).

13. McAdam, S.," Two conductor theorems," J. Algebra, 23 (1972), 239-240.

14. Sheldon, P. B.," Prime ideals in GCD-domains," Canad. J. Math., 26 (1974), 98-107.

15. Vasconcelos, W. V., " The local rings of global dimension two," Proc. Amer. Math. Soc. 35(1972),381-386.